Chapter 1

Theory of Reinforced Concrete

1.0 NOTATION

a_u	Deflection of column due to slenderness
A_c	Net area of concrete in a column cross-section
A_s	Area of steel in tension in a beam
A'_s	Area of steel in compression in a beam
A_{sb}	Area of bent shear reinforcement
A_{sc}	Area of steel in column
A_{sv}	Area of steel in vertical links
b	Width of reinforced concrete section
b_w	Width of web in a beam
c_0	Effective crack height at 'no slip' at steel
C	Internal compressive force in reinforced concrete section
d	Effective depth of tensile reinforcement
d'	Effective depth of compressive reinforcement
E_c	Modulus of elasticity of concrete
E_s	Modulus of elasticity of steel
f_c	Service stress in concrete
f_k	Characteristic strength of material
f_m	Mean strength of material from test results
f_s	Service stress in steel
f_y	Characteristic yield strength of steel
f_{cu}	Characteristic cube strength of concrete at 28 days
h	Overall depth of a concrete section
h_f	Thickness of flange in a T-beam
h_o	Initial crack height in reinforced concrete member
h_{max}	Maximum overall dimension of a rectangular concrete section
h_{min}	Minimum overall dimension of a rectangular concrete section
I	Moment of inertia
M	Applied bending moment
M'	Maximum moment of resistance of concrete section
M_f	Moment of resistance of concrete in flange
M_w	Moment of resistance of concrete in web
N	Ultimate axial load on column
p	Percentage of tensile reinforcement in a beam $= 100A_s/bd$
p	Percentage of total reinforcement in a column $= 100A_s/bh$
q	Shear flow (kN/m)
Q	First moment of area above plane of interest

r_b	Curvature of a member in bending
s	Standard deviation
S	Spacing of shear reinforcement
T	Internal tensile force in steel reinforcement
v	Shear stress in concrete (N/mm^2)
v_c	Design concrete shear stress (N/mm^2)
v_t	Shear stress in concrete due to torsion (N/mm^2)
V	Shear force in concrete section
V_c	Design concrete shear capacity
V_s	Design shear capacity of shear reinforcement
x	Depth of neutral axis from compression face
y	Distance from neutral axis
z	Depth of lever arm
α	Angle of inclination to horizontal of shear reinforcement
β	Angle of inclination to horizontal of concrete strut in truss analogy
β_a	Empirical factor governing deflection of slender columns
β_b	Ratio of redistributed moment over elastic analysis moment
β_f	Factor governing moment of resistance of concrete T-section
γ_m	Material factor
δ	Deflection of beam
ε_y	Strain at yield of steel reinforcement

1.1 INTRODUCTION

The criteria which govern the design of a structure for a particular purpose may be summarised as follows:

Fitness for purpose
Safety and reliability
Durability
Good value for money
External appearance
User comforts
Robustness.

Fitness for purpose is generally covered by the overall geometry of the structure and its components. It should be possible to have unrestricted and unhindered use of the structure for the purpose for which it is built.

Safety and reliability are assured by following the Codes of Practice for loading, materials, design, construction and fire-resistance.

Durability is taken care of by the choice of the right material for the purpose and also by bearing in mind during the design process, the requirements for proper maintenance.

Good value for money is perhaps the most important criterion. The designer should take into account not only the cost of materials but also the buildability, the time required to build, the cost of temporary structures, the cost of maintenance over a period of time and in some cases the cost of demolition/decommissioning.

External appearance of structures changes over a period of time. The designer should be aware of the effects of cracking, leaking, staining, spalling, flaking, etc. of the materials in use. The designer should make appropriate allowances to avoid the degradation of appearance.

User comforts are influenced by the vibration of the structure due to wind, road/rail traffic or vibrating machinery. Large deflections under load also cause alarm to the users. The designer should pay adequate attention to alleviation of these anticipated discomforts.

Robustness comes with the chosen structural form and is determined by the additional inherent strength of the structure as a whole to withstand accidental loadings. Collapse of one key member in the structure must not initiate global collapse. The design must foresee the 'domino effect' in the structure and avoid it by careful planning.

1.2 CHARACTERISTIC STRENGTH OF MATERIALS

The characteristic strength of a material is defined as the strength below which 1 in 20 test results are likely to fall.

The value of the characteristic strength is defined statistically by the following formula

$$f_k = f_m - 1.64s$$

where f_k = characteristic strength of material
f_m = mean strength of material from test results
1.64 is a factor which defines the 1 in 20 test results falling below f_k
s is the standard deviation.

The characteristic strength of concrete, f_{cu}, is the cube strength of concrete at 28 days.
The characteristic strength of reinforcing steel, f_y is the strength at yield.

1.3 MATERIAL FACTORS

To obtain the design strength of materials a further factor called the material factor γ_m is applied. The material factor takes into account the tolerances associated with the geometry, the variability of materials on

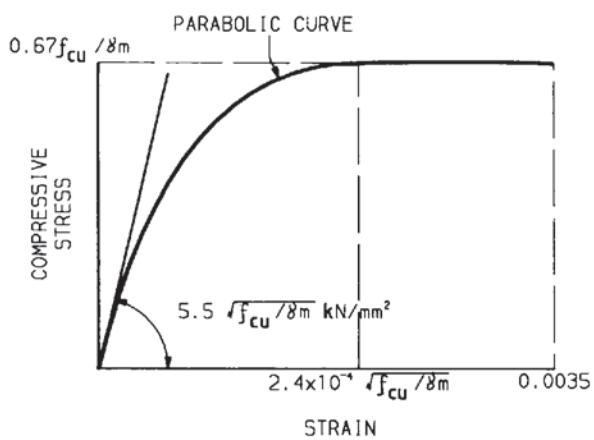
site, the inconsistency in the manufacture and curing on site and the effects of long-term degradation.
The values of γ_m for the ultimate limit state are as follows:

reinforcement	1.15
concrete in flexure or axial load	1.50
concrete in shear	1.25
bond strength in concrete	1.40
bearing stress	1.50

For exceptional loads and for localised damage, γ_m may be taken equal to 1.3 for concrete and 1.0 for reinforcement.

1.4 MATERIAL STRESS–STRAIN RELATIONSHIP

1.4.1 Short-term design stress–strain curve for normal weight concrete



SK 1/1 Short-term design stress–strain curve for normal weight concrete.

The features of this design curve are as follows:

- The *initial elastic modulus* of concrete may be defined as the initial tangent to the parabolic curve which is given by:

$$E_c = 5.5\left(\frac{f_{cu}}{\gamma_m}\right)^{\frac{1}{2}} \text{kN/mm}^2$$

- The *ultimate stress* in concrete for design purposes is defined as:

$$\sigma_u = 0.67\,\frac{f_{cu}}{\gamma_m}\ \text{N/mm}^2$$

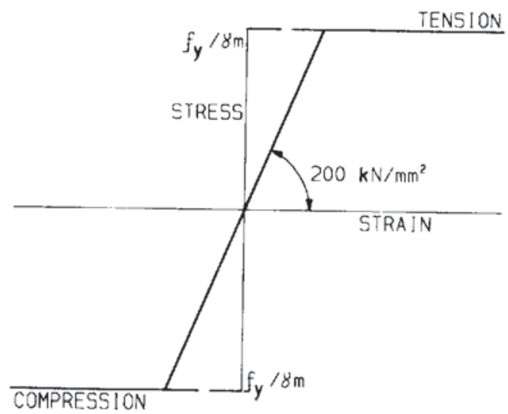
- The *ultimate strain* in concrete for design purposes is taken as 0.0035. Beyond that strain level the concrete loses its compressive stiffness.

The strain in concrete when the parabolic stress–strain relationship reaches the ultimate stress level is given by:

$$\varepsilon_y = 2.4 \times 10^{-4}\left(\frac{f_{cu}}{\gamma_m}\right)^{\frac{1}{2}}$$

Note: Concrete can withstand compressive stresses only. The tensile stress in concrete is ignored in the design.

1.4.2 Short-term design stress–strain curve for reinforcement



SK 1/2 Short-term design stress–strain curve for reinforcement.

The features of this design curve are as follows:

- The *elastic modulus* of steel reinforcement regardless of grade of steel may be assumed as 200 kN/mm^2, which is the slope of the curve up to yield.
- The *yield stress* of steel reinforcement is f_y, but for design purposes it will be taken as f_y/γ_m.
- The *stress* after yield remains constant and is represented by a constant stress line.

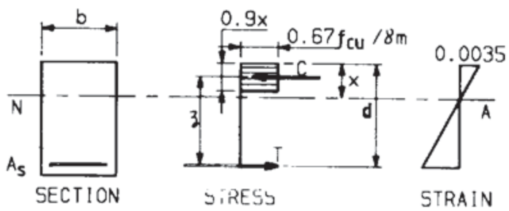
The stress–strain relationship is identical in tension and compression.

$$\varepsilon_y = 0.87\frac{f_y}{200}$$

$$= 2.0 \times 10^{-3} \quad \text{for } f_y = 460\,\text{N/mm}^2$$

1.5 DESIGN FORMULAE FOR REINFORCED CONCRETE SECTIONS

1.5.1 Singly reinforced rectangular section



SK 1/3 Stress–strain diagrams of a reinforced concrete section subject to bending moment.

Plane section remains plane.

Applied moment on the section = M; $\gamma_m = 1.5$ for concrete, 1.15 for steel.

C = compressive force in section

$$= \left(\frac{0.67}{1.5}\right) f_{cu} b(0.9x) = 0.402 f_{cu} bx$$

T = force in steel reinforcement

$$= \left(\frac{f_y}{\gamma_m}\right) A_s = 0.87 f_y A_s$$

where A_s = area of tensile steel in section

d = effective depth from outer compressive fibre to centroid of steel reinforcement.

By internal force equilibrium,

$$C = T$$

or $$0.402 f_{cu} bx = 0.87 f_y A_s$$

or $$x = 2.164\left(\frac{f_y A_s}{f_{cu} b}\right)$$

$$z = d - 0.45x = d\left(1 - 0.97\left(\frac{f_y A_s}{f_{cu} bd}\right)\right)$$

or $$\frac{z}{d} = 1 - 0.97\left(\frac{f_y A_s}{f_{cu} bd}\right)$$

or $$A_s = \left(1 - \frac{z}{d}\right)\left(\frac{f_{cu} bd}{0.97 f_y}\right)$$

$$M = 0.87 f_y A_s z = \left(\frac{0.87}{0.97}\right)\left(1 - \frac{z}{d}\right) f_{cu} bdz$$

$$= 0.90\left(1 - \frac{z}{d}\right) f_{cu} bd^2 \left(\frac{z}{d}\right)$$

$$\frac{M}{f_{cu} bd^2} = K = 0.90\left(1 - \frac{z}{d}\right)\left(\frac{z}{d}\right)$$

or $\quad \frac{z}{d} = \left[0.5 + \left(0.25 - \frac{K}{0.9}\right)^{\frac{1}{2}}\right]$

Maximum moment of resistant of concrete section is obtained for *redistribution not exceeding 10%*, when $x = d/2$.

or $\quad z = d - 0.45x = 0.775d$

Moment of resistance of concrete (maximum), M', is given by

$$M' = 0.402f_{cu}bxz$$
$$= 0.402f_{cu}b\left(\frac{d}{2}\right)(0.775d)$$
$$= 0.156f_{cu}bd^2$$

Where redistribution exceeds 10%,

$$x \leq (\beta_b - 0.4)d$$

Similarly,

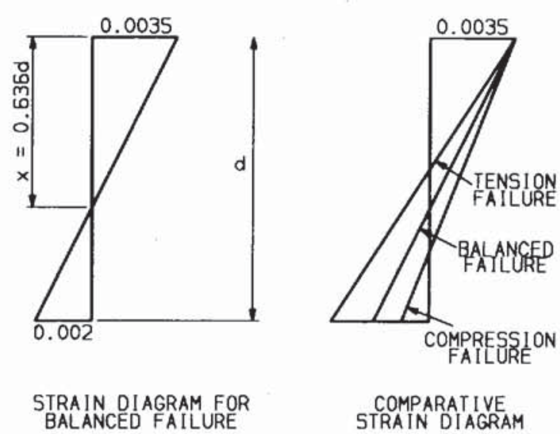
$$M' = 0.402f_{cu}bxz$$
$$= 0.402f_{cu}b(\beta_b - 0.4)d[d - 0.45(\beta_b - 0.4)d]$$
$$= [0.402(\beta_b - 0.4) - 0.18(\beta_b - 0.4)^2]f_{cu}bd^2$$

or $\quad K' = 0.402(\beta_b - 0.4) - 0.18(\beta_b - 0.4)^2$

where $\quad \beta_b = \frac{(\text{moment after redistribution})}{(\text{moment before redistribution})} < 0.9$

1.5.2 The concept of balanced design and redistribution of moments

In a singly reinforced section, if the yield strain in steel $\varepsilon_y = 0.002$ and the ultimate strain in concrete (= 0.0035) are simultaneously reached then a balanced failure condition exists.



SK 1/4 Strain diagrams of reinforced concrete section.

From strain diagram,

$$\frac{x}{(d-x)} = \frac{0.0035}{0.002} = 1.75 \quad \text{for } f_y = 460\,\text{N/mm}^2$$

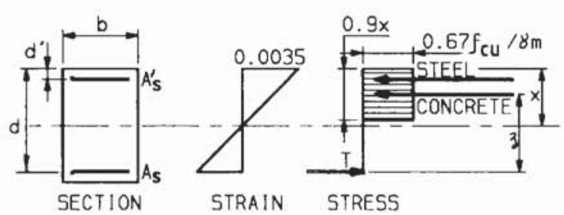
or $\quad x = 0.636d$

The Code does not allow x to be larger than $0.5d$ ensuring that the steel reaches its yield strain before the concrete reaches the ultimate strain. This is designed to allow sufficient rotational capacity in the section.

The more redistribution of moment is allowed, the more rotational capacity is needed from the section. The amount of rotation is dependent on how under-reinforced the section is, or in other words, how quickly the steel in the section reaches the yield strain before the concrete reaches the ultimate strain. To make sure that the rotational capacity exists in the section to allow redistribution, the depth of neutral axis for the design is fixed corresponding to the ratio β_b of redistribution. The compression failure is extremely brittle and must be avoided.

On the other hand, the Code has also put a limit to the minimum value of x. It has done so by limiting z to a maximum value of $0.95d$, which limits x to $0.11d$. This limitation is to avoid a very thin stress block at the ultimate state.

1.5.3 Doubly reinforced rectangular section



SK 1/5 Stress–strain diagram for doubly reinforced section.

Plane section remains plane.

The design bending moment is greater than $K'f_{cu}bd^2$, which means the concrete moment capacity is exceeded. The neutral axis is fixed by the Code depending on the amount of redistribution or $x = (\beta_b - 0.4)d \leq 0.5d$. This in turn fixes the lever arm z to concrete compression.

C = compressive force in section

= compression in concrete and compression in steel

$$= 0.9x\left(\frac{0.67f_{cu}}{\gamma_m}\right)b + \frac{A'_s f_y}{\gamma_m}$$

$$= \left(\frac{K'f_{cu}bd^2}{z}\right) + 0.87f_y A'_s$$

T = tensile force = $0.87f_y A_s$

Equating $C = T$,

$$A_s = \left(\frac{K' f_{cu} b d^2}{0.87 f_y z}\right) + A_s'$$

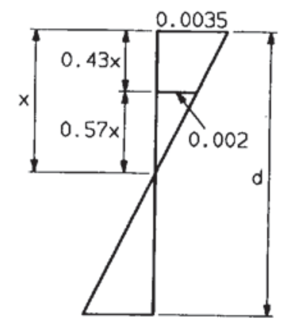
Applied moment M is equal to the moment of the internal forces. Taking moment about the centre of steel in tension,

$$M = K' f_{cu} b d^2 + 0.87 f_y A_s'(d - d')$$

$$A_s' = \frac{M - K' f_{cu} b d^2}{0.87 f_y (d - d')}$$

$$= \frac{(K - K') f_{cu} b d^2}{0.87 f_y (d - d')}$$

In the above formula it is assumed that the compressive steel will attain yield. This is true provided d' is less than or equal to $0.43x$ or the strain in the steel is at least 0.002 for $f_y = 460\,\text{N/mm}^2$. If d'/x is greater than $0.43x$, the steel stress f_s' will be proportionately modified to account for the reduced strain. Use f_s' in the equation for A_s' instead of $0.87 f_y$.



SK 1/6 Doubly reinforced beam strain diagram.

$$A_s' = \frac{(K - K') f_{cu} b d^2}{f_s'(d - d')}$$

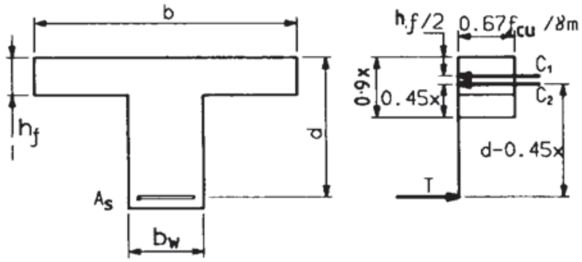
where $f_s' = \left(\frac{x - d'}{0.57x}\right) \varepsilon_y E_s$

$$\varepsilon_y = \frac{f_y}{\gamma_m E_s}$$

1.5.4 Singly reinforced flanged beams

The formulation is exactly the same as in a rectangular beam with b equal to the width of the flange provided 0.9 times the depth of the neutral axis x is less than or equal to the depth of the flange.

When $0.9x$ is greater than the depth of flange, then the following analysis will apply.



SK 1/7 Stress diagram for a flanged beam section.

C_1 = compressive force in flange only without web

$$= \left(\frac{0.67 f_{cu}}{\gamma_m}\right)(b - b_w)h_f$$

$$= 0.45 f_{cu}(b - b_w)h_f$$

C_2 = compressive force in web as in a singly reinforced beam

$$= 0.45 f_{cu} b_w (0.9x) = 0.402 f_{cu} b_w x$$

T = tension in steel

$$= \left(\frac{f_y}{\gamma_m}\right) A_s = 0.87 f_y A_s$$

The maximum allowable value of x equals $0.5d$ when the concrete moment of resistance reaches its maximum value.
Assume $x = d/2$.
Taking moment about the centre of tensile steel,

$$M' = C_1\left(d - \frac{h_f}{2}\right) + C_2(d - 0.45x)$$

$$= 0.45 f_{cu}\ (b - b_w)h_f\left(d - \frac{h_f}{2}\right) + 0.201 f_{cu} b_w d(d - 0.225d)$$

$$= f_{cu} bd^2 \left[\left(0.45\frac{h_f}{\mathrm{d}}\right)\left(1 - \frac{b_w}{b}\right)\left(1 - \frac{h_f}{2d}\right) + 0.157\,\frac{b_w}{b}\right]$$

$$= \beta_f f_{cu} bd^2$$

Values of β_f for different ratios of b/b_w and d/h_f are found in Fig. 2.1 (Chapter 2).
If the applied moment exceeds $\beta_f f_{cu} bd^2$, then compressive steel in the flange will be required.
To find the tensile steel take moment about C_1 assuming $x = d/2$.

$$M = T\left(d - \frac{h_f}{2}\right) - C_2\left(0.45x - \frac{h_f}{2}\right)$$

$$= 0.87 f_y A_s\left(d - \frac{h_f}{2}\right) - 0.1 f_{cu} b_w d(0.45d - h_f)$$

$$A_s = \frac{M + 0.1 f_{cu} b_w d(0.45d - h_f)}{0.87 f_y(d - 0.5h_f)}$$

Another approach to the design of flanged beams is presented below. When $x = h_f/0.9$, the stress block is situated entirely in the flange.

$$C = \text{compressive force} = bh_f\left(0.67\,\frac{f_{cu}}{\gamma_m}\right)$$

$$\text{lever arm} = d - \frac{h_f}{2}$$

$$M_f = 0.45 f_{cu} b h_f\left(d - \frac{h_f}{2}\right)$$

This is the flange resistance and if the applied moment exceeds this value, then the web comes into compression.
The moment to be carried by the web is M_w, when M is the applied moment.

$$M_w = M - (\text{compression in flange only outside web}) \times (\text{lever arm of flange})$$

$$= M - 0.45 f_{cu}(b - b_w)h_f\left(d - \frac{hf}{2}\right)$$

$$= M - M_f\,\frac{(b - b_w)}{b}$$

$$= M - M_f\left(1 - \frac{b_w}{b}\right)$$

Find M_f, and if M_f is less than M, then find M_w by the above formula. Design for M_w as for a rectangular beam with width equal to b_w. Find A_{s1} for M_f and A_{s2} for M_w.

Total $A_s = A_{s1} + A_{s2}$

$$A_{s1} = \frac{0.45 f_{cu}(b - b_w)h_f}{0.87 f_y}$$

$$A_{s2} = \frac{M_w}{0.87 f_y z}$$

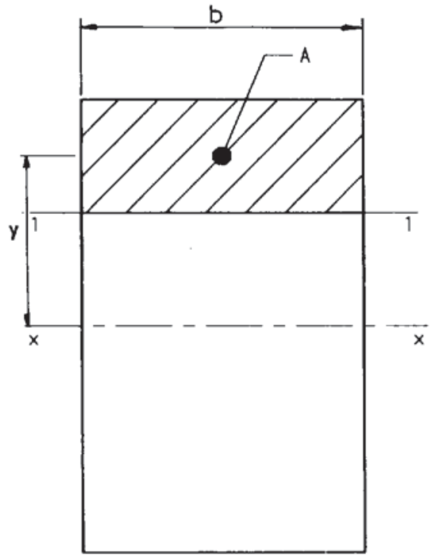
when $K = \dfrac{M_w}{f_{cu} b_w d^2}$

and $z = d\left[0.5 + \left(0.25 - \dfrac{K}{0.9}\right)^{\frac{1}{2}}\right]$

Note: The design against M_w may follow Section 1.5.3, which means that the flanged section may be doubly reinforced, if required.

1.6 ULTIMATE LIMIT STATE – SHEAR

The horizontal shear stress in a homogeneous, isotropic, uncracked beam is given by the classical expression:



SK 1/8 Calculation of shear stress in a homogeneous section.

$$v = \frac{VQ}{Ib} \text{ N/mm}^2$$

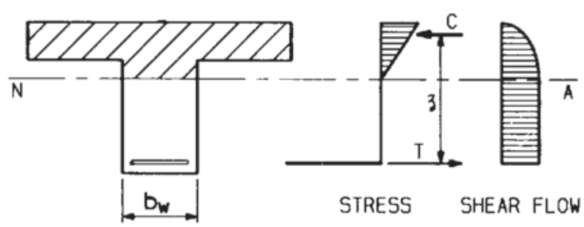
where Q = first moment of area above line 1–1 = Ay
I = second moment of area of the section about x–x
b = width of section on line 1–1
v = shear stress at line 1–1
V = shear at the section
A = area of section above line 1–1
y = distance of the centroid of area A from the neutral axis.

$$\text{Shear flow, } q = \frac{VQ}{I}$$

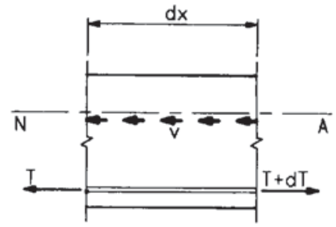
In a concrete beam where concrete is ignored, the horizontal shear stress can be included in the expression below for horizontal equilibrium under the neutral axis.

$$\mathrm{d}T = vb_w\mathrm{d}x$$

The shear flow in the tension zone of concrete will be constant because the concrete is ignored.



SK 1/9 Calculation of shear stress.



SK 1/10 Shear stress in a reinforced concrete section.

From the above expression,

$$v = \left(\frac{1}{b_w}\right)\left(\frac{dT}{dx}\right)$$

$$\frac{M}{z} = T$$

or $$\frac{dM}{z} = dT$$

$$\therefore \quad v = \left(\frac{1}{b_w}\right)\left(\frac{1}{z}\right)\left(\frac{dM}{dx}\right)$$

$$V = \frac{dM}{dx}$$

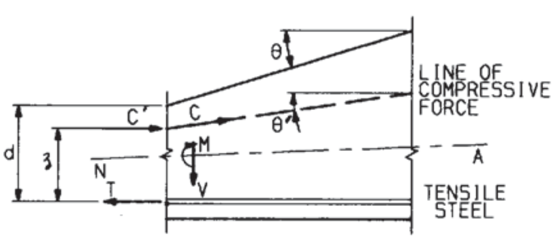
$$\therefore \quad v = \frac{V}{b_w z}$$

For convenience in the ultimate limit state the Code shear stress index is taken as:

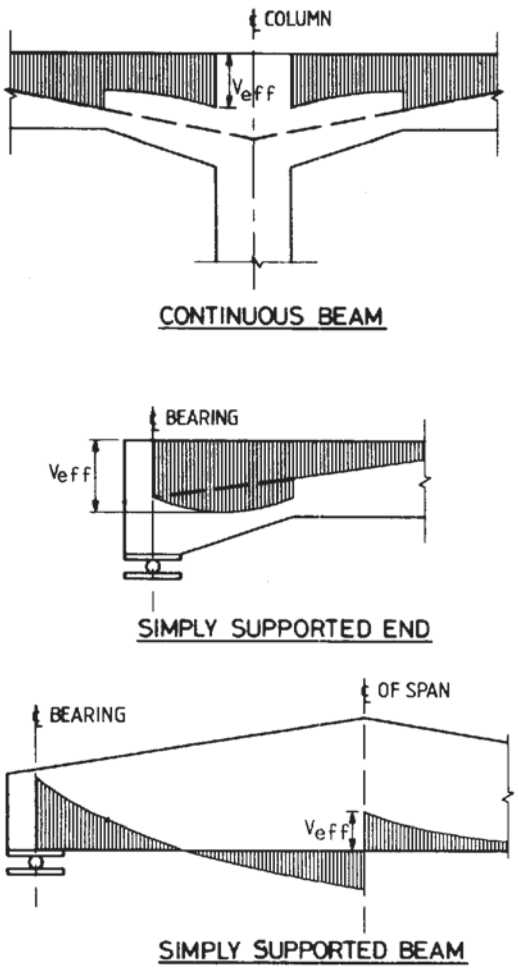
$$v = \frac{V}{b_w d}$$

Effective shear in haunched beams

$$V_{eff} = V - C \sin\theta'$$
$$= V - C' \tan\theta'$$
$$= V - \frac{M}{z}\tan\theta'$$



SK 1/11 Effective shear force in a beam with variable depth.



SK 1/12 Effective shear force diagrams for beams with variable depth.

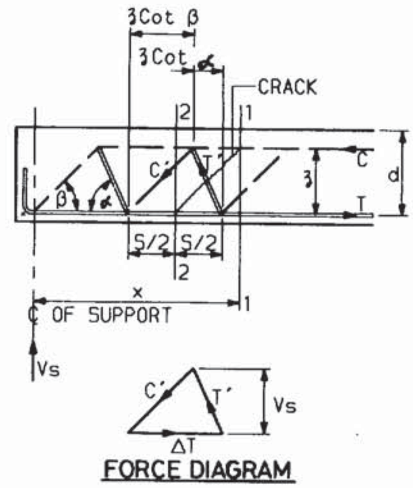
1.6.1 Principle of 'design concrete shear stress'

Shear is resisted in concrete beams by the combined action of the following:

- Shear resistance of concrete in compression zone. Dowel force in tension bars across a crack. Aggregate interlocking across the inclined crack in tension zone.
- The Code formula takes into account the dowel force of the tensile steel and the formula is essentially of empirical nature arrived at from test results.

Shear reinforcement – truss analogy

V_s = shear force to be resisted by reinforcement



SK 1/13 Truss analogy of shear reinforcement.

From the force diagram,

$$V_s = C' \sin\beta = T' \sin\alpha$$

where C' = concrete strut force

T' = tensile force in shear reinforcement

= resultant of all forces in shear reinforcement within the spacing S.

From geometry:

$$S = z(\cot\alpha + \cot\beta)$$

$$\frac{T'}{S}\sin\alpha = \frac{V_s}{S} = \frac{V_s}{z(\cot\alpha + \cot\beta)}$$

or $$\frac{T'}{S} = \frac{V_s}{z\sin\alpha(\cot\alpha + \cot\beta)}$$

or $$\frac{0.87f_yA_{sb}}{S} = \frac{V_s}{z\sin\alpha(\cot\alpha + \cot\beta)}$$

or $$V_s = 0.87f_yA_{sb}(\cos\alpha + \sin\alpha\cot\beta)\left(\frac{z}{S}\right)$$

The Code uses $(d - d')$ in place of z in the formula.

When vertical stirrups are used and the concrete struts are assumed inclined at 45° to horizontal then $\alpha = 90°$ and $\beta = 45°$.

$$\therefore\ V_s = 0.87f_{yv}A_v\left(\frac{z}{S}\right)$$

$$v_s = \frac{V_s}{bz} = \frac{0.87f_{yv}A_v}{bS}$$

$$v_s \geq (v - v_c)$$

or $$A_{sv} \geq \frac{bS(v - v_c)}{0.87f_{yv}}$$

Note: The assumptions in the above truss analogy are:

- Bond forces are sustained along the length of the beam where shear reinforcement is required.
- The lever arm z is assumed constant over the section with variable moment producing the shear to be resisted. The diagonal compressive stress in concrete struts calculated from the analogy is equal to $v_s/[\sin^2\beta(\cot\alpha + \cot\beta)]$ is sustainable.

1.6.2 Additional tensile steel in conjunction with shear reinforcement

Referring to SK 1/13, assume that a diagonal crack forms in concrete when the shear force exceeds the concrete shear capacity, V_c. Assume that the ultimate shear force is $V_u = V_c + V_s$, where V_s is resisted by shear reinforcement.
Sections 1–1 and 2–2 are taken at two ends of a diagonal crack.
Consider the free body diagram of internal forces.
Assume that the tensile force requirement at section 2 is also divided in two parts. When the shear becomes V_c, the tensile force required is T_c, and for V_s it is T_s.
Assume that the moment at section 2 is M_{2c} corresponding to V_c and M_{2s} corresponding to V_s.

$$\therefore V_u = V_c + V_s$$
$$T_u = T_c + T_s$$
$$M_{2u} = M_{2c} + M_{2s}$$

T' is the tensile force in the shear reinforcement.

Initially assume a shear of V_c on section.
Taking moment about the concrete compressive force C at section 1–1,

$$M_{1c} = V_c x = M_{2c} + V_c z \cot\beta = T_c z$$

or $$T_c = \frac{M_{2c}}{z} + V_c \cot\beta$$

Note: At this stage $T' = 0$.

Next assume a shear of V_s on section.
Taking moment about the concrete compressive force C at section 1–1,

$$M_{1s} = V_s x = M_{2s} + V_s z \cot\beta = T_s z + \left(\frac{S}{2}\right) T' \sin\alpha$$

or $$T_s = \frac{M_{2s}}{z} + V_s \cot\beta - \left(\frac{S}{2z}\right) T' \sin\alpha$$

Substituting:

$$T' \sin\alpha = V_s \quad \text{and} \quad S = z(\cot\alpha + \cot\beta)$$

we get:

$$T_s = \frac{M_{2s}}{z} + \left(\frac{V_s}{2}\right)(\cot\beta - \cot\alpha)$$

$$T_u = T_c + T_s = \left(\frac{M_{2c} + M_{2s}}{z}\right) + V_c \cot\beta + \left(\frac{V_s}{2}\right)(\cot\beta - \cot\alpha)$$

$$= \frac{M_u}{z} + V_c \cot\beta + \left(\frac{V_s}{2}\right)(\cot\beta - \cot\alpha)$$

This demonstrates quite clearly that when diagonal cracks form in concrete due to shear exceeding V_c, additional tensile steel will be required over and above M_u/z.

This requirement is not explicitly covered in the Code. The rules of curtailment of reinforcement are deemed to satisfy this requirement.

Note: At locations of very high shear this additional requirement should be checked.

1.7 SERVICEABILITY LIMIT STATE – CRACK WIDTH

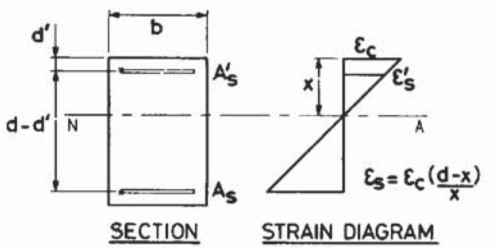
The basic assumptions to find crack width for flexure are summarised below:

(1) Plane section remains plane before and after bending.
(2) Concrete compressive stress diagram is linear and triangular. The stress is directly proportional to strain.
(3) The short-term Young's modulus of concrete may be used.
(4) The steel reinforcement does not go beyond yield.
(5) The loading is at serviceability limit state.
(6) Effective crack height at 'no slip' is C_o, which is the minimum cover to reinforcement.
(7) Mean crack spacing is 1.5 C_o.
(8) Initial crack height h_o is up to the neutral axis and the maximum crack width is a function of the ratio C_o/h_o to take into account slip and internal cracks in concrete.
(9) Stiffening effect of tension in concrete is allowed for by an empirical term.
(10) The equation for the determination of maximum crack width is empirical and is a close fit of test results. It is anticipated that 1 in 5 results will exceed the prediction by the formula.

For the formula and its application, see Chapter 2.

1.8 SERVICEABILITY LIMIT STATE – DEFLECTION

Calculate moments at service load without redistribution.



SK 1/14 Doubly reinforced section.

The assumptions for the analysis of the section are similar to assumptions (1) to (5) in Section 1.7.

Find the depth of neutral axis, x, and find the stresses in concrete, f_c, and the stress in steel, f_s, by following Step 25 of worked example in Chapter 2. This method ignores the concrete under the neutral axis.

$$\text{Curvature} = \frac{1}{r_b} = \frac{f_c}{xE_c} = \frac{f_s}{(d-x)E_s}$$

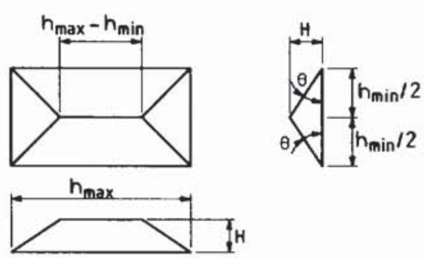
Alternatively, calculate x as previously and then allow a tensile stress in concrete up to $1\,\text{N/mm}^2$ short-term and $0.55\,\text{N/mm}^2$ long-term.

$$\text{Deflection, } \delta = Kl^2\left(\frac{1}{r_b}\right)$$

where K depends on the shape of the bending moment diagram.

BS 8110: Part 2: 1985 [1] in Table 3.1 gives different values of K for various loadings and support conditions. The principle of superposition may be used to combine different types of loading.

1.9 ULTIMATE LIMIT STATE – TORSION



SK 1/15 Membrane analogy for torsion.

By principles of membrane analogy it is known that 2 times the volume included between the surface of a deflected membrane and the plane of its outline is equal to the torque in a twisted member.

Applying this analogy to a rectangular section gives a pyramidal deflected membrane.

$$\text{Volume of pyramid} = \tfrac{1}{3}\,h_{min}^2\,H + \tfrac{1}{2}\,h_{min}(h_{max} - h_{min})H$$
$$= \left(\frac{h_{min}H}{2}\right)\left(h_{max} - \frac{h_{min}}{3}\right)$$

By membrane theory it is known that the torsional shear stress is the slope of the angle of the deflected membrane.

$$\tan\theta = \frac{H}{(h_{min}/2)} = v_t$$

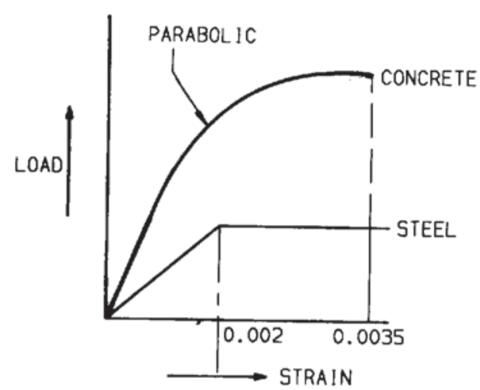
$$T = 2 \times \text{volume of pyramid} = h_{min}H\left(h_{max} - \frac{h_{min}}{3}\right)$$

Substituting $H = v_t(h_{min}/2)$,

$$T = \left(\frac{v_t h_{min}^2}{2}\right)\left(h_{max} - \frac{h_{min}}{3}\right)$$

or $$v_t = \frac{2T}{h_{min}^2\left(h_{max} - \frac{h_{min}}{3}\right)}$$

1.10 ULTIMATE LIMIT STATE – COLUMNS



SK 1/16 Typical load–strain curve of a column.

1.10.1 Axial load capacity of columns

Taking creep and shrinkage of concrete into consideration it is difficult to predict the actual stresses in a short concrete column subjected to service axial load in the elastic range, because initial compressive stress from concrete gets transferred to the steel due to creep of concrete. But at the ultimate load stage it is easier to predict the ultimate load-carrying capacity because the concrete ultimate strain of 0.0035 is much higher than steel

yield strain. Hence, the steel reaches its *ultimate* load-carrying capacity long before concrete gets there.
The ultimate load-carrying capacity of a short reinforced column may be written as,

$$N_{uz} = \left(\frac{0.67 f_{cu}}{\gamma_m}\right) A_c + \left(\frac{f_y}{\gamma_m}\right) A_{sc}$$
$$= 0.45 f_{cu} A_c + 0.87 f_y A_{sc}$$

where A_c = net concrete cross-sectional area
A_{sc} = area of compressive steel reinforcement.

The Code equations allow for a nominal eccentricity and the formulae are changed to one of the following depending on application:

$$N = 0.4 f_{cu} A_c + 0.75 f_y A_{sc}$$

for a column with nominal eccentricity of load – meaning a column with no design moments and eccentric loads. The eccentricity is allowed for the constructional tolerances.

$$N = 0.35 f_{cu} A_c + 0.67 f_y A_{sc}$$

for a column supporting an approximately symmetrical arrangement of beams. The spans of the beams on either side of the column should not differ by more than 15%. To allow for a certain eccentricity of loading due to the variations in spans and the location and disposition of live loadings on spans, the equation has been modified.

1.10.2 Axial load capacity of slender columns

The strength of a slender column depends on:

(1) Effective height-to-width ratio, where the effective height depends on the rotational end restraints and the lateral restraints by bracing.
(2) The flexural rigidity of the column section which determines the Euler critical buckling load.
(3) The duration of loading which influences the strength and deflections due to creep.

The Code uses the 'Moment Magnifier Method', whereby the effect of slenderness is transferred into an equivalent deflection and an additional moment given by the product of this deflection and the applied direct load.

$$M_{add} = N a_u$$

where $a_u = \beta_a K h$

$$\beta_a = \frac{1}{2000}\left(\frac{l_e}{b'}\right)^2$$

$$K = \left(\frac{N_{uz} - N}{N_{uz} - N_{bal}}\right) \leq 1$$

$N_{bal} = 0.25 f_{cu} bd$ approximately (see Section 1.10.4.1).
h = dimension of column in the plane of bending considered
b' = shorter dimension of column for uniaxial bending, or
= dimension of column in the plane of bending considered for significant biaxial bending
l_e = effective length of column in the plane of bending considered.

(See Chapter 4 for further explanation.)

1.10.3 Axial load and moment on column

The assumptions for the analysis of the section are exactly the same as in the case of beams and the analysis depends on strain compatibility. The design is usually carried out by the use of published charts which have been derived using the following assumptions:

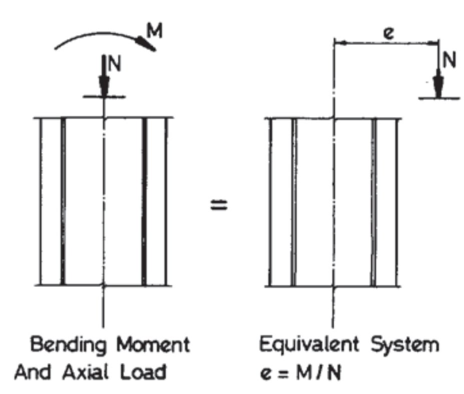
(1) Plane section remains plane or the strain compatibility is assumed.
(2) The concrete stress block is assumed rectangular-parabolic.
(3) The stress–strain curve for steel is bilinear.

To use the charts to find the total area of steel required, the following parameters are required: f_{cu}, f_y, N/bh, M/bh^2 and the d/h ratio. (See Chapter 4 for further explanation.)

1.10.4 Column interaction diagrams

1.10.4.1 Rectangular section

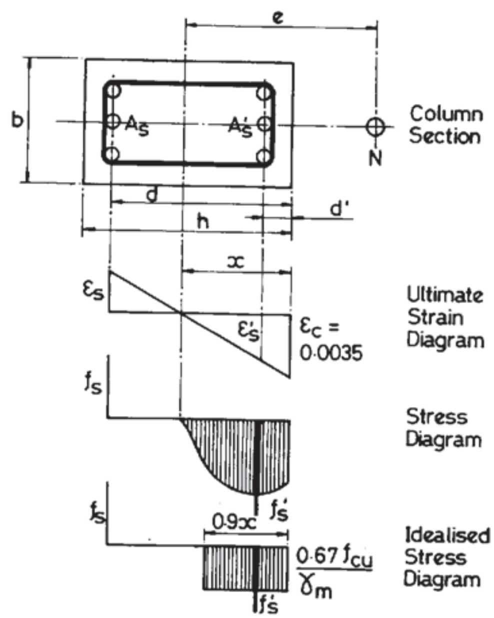
If column charts are not available and a hand calculation is required or, where the column size and reinforcement are known and the column load-carrying capacity with variable eccentricity is required for assessment purposes, the following design procedure may be followed. The interaction diagram of a column with known areas of steel will illustrate the ultimate



SK 1/17 Elevation of a column.

load-carrying capacity of the column section subjected to uniaxial bending moment.

N = applied ultimate direct load
M = applied ultimate coacting bending moment
$e = M/N$ = eccentricity of direct load
C_c = resistance of concrete in compression
C_s = resistance of steel in compression
T = resistance of steel in tension



SK 1/18 Stress and strain diagrams.

Balanced failure

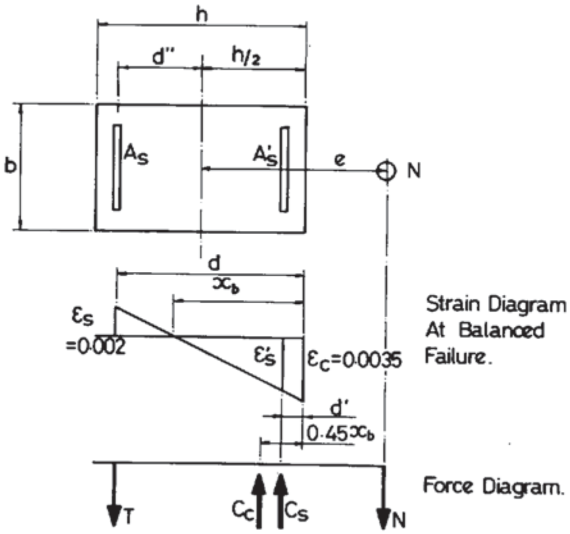
A *balanced failure* occurs when the tension steel just reaches yield at the same time as the extreme compression fibre in concrete reaches the ultimate strain of 0.0035.

f_y = characteristic yield strength of steel
E_s = modulus of elasticity of steel = $200\,\text{kN/mm}^2$
x_b = depth of neutral axis at 'balanced failure'

From the strain diagram:

$$\frac{0.0035}{x_b} = \frac{f_y/E_s\gamma_m}{(d - x_b)}$$

assuming $f_y = 460\,\text{N/mm}^2$
$\gamma_m = 1.15$ for steel
$E_s = 200\,\text{kN/mm}^2$



SK 1/19 Strain and force diagram at balanced failure.

the above equation reduces to:

$$\frac{0.0035}{x_b} = \frac{0.002}{(d - x_b)}$$

or $\quad x_b = 0.636d$

$$N_b = \text{ultimate load at balanced failure}$$
$$= C_s + C_c - T$$
$$= A_s' \,(0.87f_y) + \left(\frac{0.67f_{cu}}{\gamma_m}\right)0.9x_b b - A_s(0.87f_y)$$

For a symmetrically reinforced section where $A_s = A_s'$, and the compression reinforcement reaches yield strain, the terms for C_s and T cancel each other out.

$$\therefore \quad N_b = 0.256f_{cu}bd$$

The strain in compression steel is governed by the value of d'.
The yield strain in compression steel is $f_y/E_s\gamma_m$.
$E_s = 200\,\text{kN/mm}^2$ and $\gamma_m = 1.15$.
For Grade $460\,\text{N/mm}^2$ steel, this yield strain = 0.002.
At 'balanced failure' condition:

$$\frac{0.0035}{x_b} = \frac{\varepsilon_s'}{(x_b - d')}$$

Assuming $\varepsilon_s' = 0.002$ and $x_b = 0.636d$, the maximum value of d' to produce yield strain in compression steel is given by:

$$d' = 0.273d$$

or $\quad \dfrac{d}{h} = 0.78$ minimum to produce yield strain in compression reinforcement at 'balanced failure'.

For a symmetrically reinforced section,

M_b = moment to produce balanced failure

$d/h = k$

$p = 100A_s/bh$ = percentage of reinforcement with respect to tension reinforcement only

$f_y = 460\,\text{N/mm}^2$

$d'/h = (1 - k)$

$d''/h = (k - 0.5)$

Taking moment about the centroid of the section,

for $k \geq 0.78$

$$M_b = 0.402 f_{cu} x_b b(0.5h - 0.45x_b) + 0.87A_s f_y h(k - 0.5) + 0.87A_s f_y h(k - 0.5)$$

for $k < 0.78$

$$M_b = 0.402 f_{cu} x_b b(0.5h - 0.45x_b) + 0.87A_s f_y h(k - 0.5) + A_s f_s h(k - 0.5)$$

where $f_s = \varepsilon_s E_s$ and $\varepsilon_s = \dfrac{0.0035(x_b - d')}{x_b}$

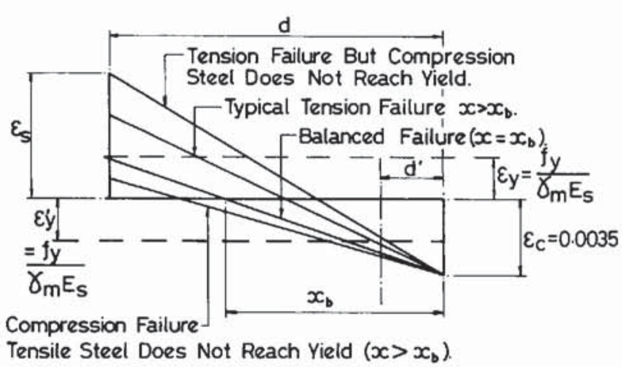
for $k \geq 0.78$

$$\frac{M_b}{bh^2} = 0.256 f_{cu} k(0.5 - 0.286k) + 8p(k - 0.5)$$

for $k < 0.78$

$$\frac{M_b}{bh^2} = 0.256 f_{cu} k(0.5 - 0.286k) + 4p(k - 0.5) + 11p(1.636k - 1)\frac{(k - 0.5)}{k}$$

Note: In the above equations use $0.5p$ instead of p if p is the percentage of total reinforcement in the column.



SK 1/20 Strain diagram of column for various types of failure.

Tension failure

When $N < N_b$ and $f_s = f_y/\gamma_m$, a tension failure condition will apply. The column behaves more like a beam in this condition.

$$N = C_c + C_s - T$$

Assuming symmetrical reinforcement and yield strain in both tension and compression steel,

$$N = C_c = 0.402 f_{cu} bx$$

or $$x = \frac{N}{0.402 f_{cu} b}$$

Check that

$$\varepsilon_s' = \frac{0.0035(x - d')}{x} \geq 0.002 \qquad \text{for } f_y = 460\,\text{N/mm}^2$$

or $$x \geq 2.331h(1 - k)$$

This ensures that the compression steel has reached yield.
For a symmetrically reinforced rectangular section,

$$\frac{x}{h} = 2.4875\left(\frac{N}{f_{cu}bh}\right) \qquad \text{for } \frac{x}{h} \geq 2.331(1 - k)$$

$$= 0.5[(B^2 + 4C)^{\frac{1}{2}} - B] \qquad \text{for } \frac{x}{h} < 2.331(1 - k)$$

where $$B = 7.463\left(\frac{p}{f_{cu}}\right) - 2.4875\left(\frac{N}{f_{cu}bh}\right)$$

$$C = 17.413(1 - k)\left(\frac{p}{f_{cu}}\right)$$

Taking moment about the centroid of the section,

for $\frac{x}{h} \geq 2.331(1 - k)$

$$M = 0.402 f_{cu} xb(0.5h - 0.45x) + 0.87A_s f_y h(k - 0.5) + 0.87A_s f_y h(k - 0.5)$$

for $\frac{x}{h} < 2.331(1 - k)$

$$M = 0.402 f_{cu} xb(0.5h - 0.45x) + 0.87A_s f_y h(k - 0.5) + A_s f_s h(k - 0.5)$$

for $\frac{x}{h} \geq 2.331(1 - k)$

$$\frac{M}{f_{cu}bh^2} = 0.402\left(\frac{x}{h}\right)\left(0.5 - 0.45\left(\frac{x}{h}\right)\right) + \frac{8p(k - 0.5)}{f_{cu}}$$

for $\frac{x}{h} < 2.331(1-k)$

$$\frac{M}{f_{cu}bh^2} = 0.402\left(\frac{x}{h}\right)\left[0.5 - 0.45\left(\frac{x}{h}\right)\right] + \frac{11p(k - 0.5)}{f_{cu}} - \frac{7p[(1 - k)/(x/h)](k - 0.5)}{f_{cu}}$$

Compression failure

When $N > N_b$, a compression failure condition applies.

$$N = C_c + C_s - T$$

Assuming symmetrical reinforcement and yield strain in both tension and compression steel,

$$N = C_c = 0.402 f_{cu} bx$$

or $$\frac{x}{h} = \frac{N}{0.402 f_{cu} bh} = 2.4875\left(\frac{N}{f_{cu}bh}\right)$$

For the tensile steel to be at yield,

$$\varepsilon_s \geq 0.002 \qquad \text{for } f_y = 460\,\text{N/mm}^2$$

or $$0.0035\frac{(d - x)}{x} \geq 0.002$$

or $$\frac{x}{h} \leq 0.636k$$

For both tensile steel and compression steel to be at yield,

$$0.636k \geq \frac{x}{h} \geq 2.331(1 - k)$$

When compression steel does not yield,

for $0.636k \geq \frac{x}{h} < 2.331(1-k)$,

$$\frac{x}{h} = 0.5[(B^2 + 4C)^{\frac{1}{2}} - B]$$

where $$B = 7.463\left(\frac{p}{f_{cu}}\right) - 2.4875\left(\frac{N}{f_{cu}bh}\right)$$

$$C = 17.413(1 - k)\left(\frac{p}{f_{cu}}\right)$$

When tension steel does not yield,

for $0.636k < \frac{x}{h} \geq 2.331(1-k)$,

$$\frac{x}{h} = 0.5[(B^2 + 4C)^{\frac{1}{2}} - B]$$

where $B = 27.363\left(\frac{p}{f_{cu}}\right) - 2.4875\left(\frac{N}{f_{cu}bh}\right)$

$$C = 17.413k\left(\frac{p}{f_{cu}}\right)$$

When tension steel and compression steel do not yield,

for $0.636k < \frac{x}{h} < 2.331(1-k)$,

$$\frac{x}{h} = 0.5[(B^2 + 4C)^{\frac{1}{2}} - B]$$

where $B = 34.826\left(\frac{p}{f_{cu}}\right) - 2.4875\left(\frac{N}{f_{cu}bh}\right)$

$$C = 17.413\left(\frac{p}{f_{cu}}\right)$$

Taking moment about the centroid of section.
For both tension and compression steel going into yield,

$$M = 0.402f_{cu}xb(0.5h - 0.45x) + 0.87A_sf_yh(k - 0.5) + 0.87A_sf_yh(k - 0.5)$$

For tension steel only going into yield,

$$M = 0.402f_{cu}xb(0.5h - 0.45x) + 0.87A_sf_yh(k - 0.5) + A_sf_s'h(k - 0.5)$$

For compression steel only going into yield,

$$M = 0.402f_{cu}xb(0.5h - 0.45x) + A_sf_sh(k - 0.5) + 0.87A_sf_yh(k - 0.5)$$

For both tension and compression steel not going into yield,

$$M = 0.402f_{cu}xb(0.5h - 0.45x) + A_sf_sh(k - 0.5) + A_sf_s'h(k - 0.5)$$

for $0.636k \geq \frac{x}{h} \geq 2.331(1-k)$

$$\frac{M}{f_{cu}bh^2} = 0.402\left(\frac{x}{h}\right)\left[0.5 - 0.45\left(\frac{x}{h}\right)\right] + \frac{8p(k - 0.5)}{f_{cu}}$$

for $0.636k \geq \frac{x}{h} < 2.331(1-k)$

$$\frac{M}{f_{cu}bh^2} = 0.402\left(\frac{x}{h}\right)\left[0.5 - 0.45\left(\frac{x}{h}\right)\right] + \left[\frac{11p(k - 0.5)}{f_{cu}}\right] - \left\{\frac{7p[(1 - k)/(x/h)]\ (k - 0.5)}{f_{cu}}\right\}$$

for $0.636k < \frac{x}{h} \geq 2.331(1-k)$

$$\frac{M}{f_{cu}bh^2} = 0.402\left(\frac{x}{h}\right)\left[0.5 - 0.45\left(\frac{x}{h}\right)\right] + \left[\frac{7pk(k-0.5)}{(x/h)f_{cu}}\right] - \left[\frac{3p(k-0.5)}{f_{cu}}\right]$$

for $0.636k < \frac{x}{h} < 2.331\ (1-k)$.

$$\frac{M}{f_{cu}bh^2} = 0.402\left(\frac{x}{h}\right)\left[0.5 - 0.45\left(\frac{x}{h}\right)\right] + \frac{7p(2k-1)(k-0.5)}{(x/h)f_{cu}}$$

Note: In all above equations use $0.5p$ instead of p if p is the percentage of total reinforcement in the column.

Symmetrical rectangular column – interaction tables for columns subject to uniaxial bending and direct load

Tables 11.8 to 11.17 (Chapter 11) have been prepared by solving by iteration the following equations. Assuming equal reinforcement in each face of the rectangular section and assuming that all steel reinforcement is in tension,

b = width of rectangular section
d_1 = depth of the top layer of reinforcement near the compression face from the compression face
d = depth of the bottom layer of reinforcement near the tension face i.e. the effective depth
h = overall depth of section
e = M/N = eccentricity of load from the centre of the rectangular section, assuming that the concentrated load N always acts at the centre of the rectangular section
f_{s1} = stress in steel at depth d_1
f_s = stress in steel at depth d
A_{s1} = $A_s/2$, i.e. equal reinforcement on each face
p = percentage of reinforcement in column = $100A_s/bh$
x = depth of neutral axis from compression face
k = d/h
$d_1/h = 1 - k$
$0.87f_y = 0.87 \times 460 = 400\ \text{N/mm}^2$
$f_{s1} = E_s\varepsilon_{s1} = 200 \times 10^3 \times 0.0035 \times (d_1 - x)/x = 700\ (d_1 - x)/x \leq |400|\ \text{N/mm}^2$
$f_s = 700\ (d - x)/x \leq |400|\ \text{N/mm}^2$
A_c = area of concrete in compression = $0.9bx = 0.9bh(x/h)$
$N = (0.67f_{cu}/\gamma_m)A_c - (A_s/2)(f_{s1} + f_s)$

$$\frac{N}{bh} = 0.402f_{cu}\left(\frac{x}{h}\right) - \left(\frac{p}{200}\right)(f_{s1} + f_s)$$

Taking moment about the centre of the rectangular section,

$$Ne = \left(\frac{0.67f_{cu}}{\gamma_m}\right)A_c\left(\frac{h}{2} - 0.45x\right) + \left(\frac{A_s}{2}\right)\left(\frac{h}{2} - (1-k)h\right)(f_s - f_{s1})$$

$$= 0.402f_{cu}\left(\frac{x}{h}\right)bh^2\left(0.5 - 0.45\left(\frac{x}{h}\right)\right) + bh^2\left(\frac{p}{200}\right)(k - 0.5)(f_s - f_{s1})$$

Dividing Ne/bh^2 by N/bh we get ε/h.

$$\therefore \quad \frac{e}{h} = \frac{0.402f_{cu}\left(\frac{x}{h}\right)\left(0.5 - 0.45\left(\frac{x}{h}\right)\right) + \left(\frac{p}{200}\right)(k - 0.5)(f_s - f_{s1})}{0.402f_{cu}\left(\frac{x}{h}\right) - \left(\frac{p}{200}\right)(f_{s1} + f_s)}$$

$$f_{s1} = \frac{700\left(\frac{d_1}{h} - \frac{x}{h}\right)}{\frac{x}{h}}$$

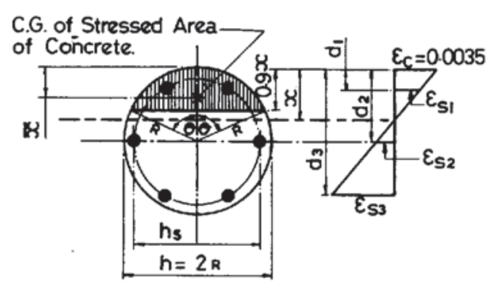
$$\text{or} \quad f_{s1} = \frac{700\left(1 - k - \frac{x}{h}\right)}{\frac{x}{h}} \leq |400|\ \text{N/mm}^2$$

$$f_s = \frac{700\left(k - \frac{x}{h}\right)}{\frac{x}{h}} \leq |400|\ \text{N/mm}^2$$

For a range of values of f_{cu}, k and p, the above equations can be solved for different values of e/h. Tables 11.8 to 11.17 give N/bh for different values of e/h.

Note: The above equations are valid up to $x = 1.111h$.

1.10.4.2 Circular section



SK 1/21 Circular column – strain diagram.

R = radius of circular section
A_c = area of equivalent uniform stressed section of concrete bounded by a line at $0.9x$ from compression face
$\bar{x}$ = centroid of stressed section of concrete

x = depth of neutral axis from compression face
e = eccentricity of applied load = M/N
N = applied direct load at centre of section
M = applied equivalent uniaxial moment
p = percentage of reinforcement = $100A_s/\pi R^2$
k = h_s/h
2θ = angle to the corner of equivalent uniformly stressed area subtended at the centre of section, or the angle subtended to the line at $0.9x$ from compression face
A_s = total area of steel in six bars
A_c = $R^2(\theta - \sin\theta\cos\theta)$

$$\bar{x} = R\left[1 - \frac{2\sin^3\theta}{3(\theta - \sin\theta\cos\theta)}\right]$$

First layer of steel is at d_1, second layer of steel is at d_2, and third layer is at d_3.

Note: If x is greater than d_1, d_2, or d_3 then the corresponding steel is in compression.

$$N = \left(\frac{0.67f_{cu}}{\gamma_m}\right)A_c - \left(\frac{A_s}{3}\right)(f_{s1} + f_{s2} + f_{s3})$$

$$= 0.45R^2 f_{cu}(\theta - \sin\theta\cos\theta) - \left(\frac{p\pi R^2}{300}\right)(f_{s1} + f_{s2} + f_{s3})$$

$$\frac{N}{R^2} = 0.45f_{cu}(\theta - \sin\theta\cos\theta) - \left(\frac{p\pi}{300}\right)(f_{s1} + f_{s2} + f_{s3})$$

$$f_{s1} = 0.0035E_s\left(\frac{d_1 - x}{x}\right) \leq |400|\ \text{N/mm}^2$$

$$\text{or}\quad f_{s1} = \frac{700\left(\frac{d_1}{R} - \frac{x}{R}\right)}{\frac{x}{R}} \leq |400|\ \text{N/mm}^2$$

$$f_{s2} = \frac{700\left(\frac{d_2}{R} - \frac{x}{R}\right)}{\frac{x}{R}} \leq |400|\ \text{N/mm}^2$$

$$f_{s3} = \frac{700\left(\frac{d_3}{R} - \frac{x}{R}\right)}{\frac{x}{R}} \leq |400|\ \text{N/mm}^2$$

$$\frac{d_1}{R} = 1 - k\cos 30^\circ \qquad \frac{d_2}{R} = 1 \qquad \frac{d_3}{R} = 1 + k\cos 30^\circ$$

$$\frac{x}{R} = \left(\frac{1}{0.9}\right)(1 - \cos\theta) = 1.11\ (1 - \cos\theta)$$

$$\frac{\bar{x}}{R} = 1 - \frac{2\sin^3\theta}{3(\theta - \sin\theta\cos\theta)}$$

Taking moment about the centre of section assuming that the applied load N is always at the centre of section,

$$Ne = 0.45R^2 f_{cu}(\theta - \sin\theta\cos\theta)(R - \bar{x}) + \left(\frac{A_s}{3}\right)(kR\sin 60°)(f_{s3} - f_{s1})$$

$$= 0.45R^3 f_{cu}(\theta - \sin\theta\cos\theta)\left(1 - \frac{\bar{x}}{R}\right) + \left(\frac{p\pi R^3}{300}\right)(k\sin 60°)(f_{s3} - f_{s1})$$

Dividing Ne/R^3 by N/R^2 we get e/R.

$$\frac{e}{R} = \frac{0.45f_{cu}(\theta - \sin\theta\cos\theta)\left(1 - \frac{\bar{x}}{R}\right) + \left(\frac{p\pi}{300}\right)(k\sin 60°)(f_{s3} - f_{s1})}{0.45f_{cu}(\theta - \sin\theta\cos\theta) - \left(\frac{p\pi}{300}\right)(f_{s1} + f_{s2} + f_{s3})}$$

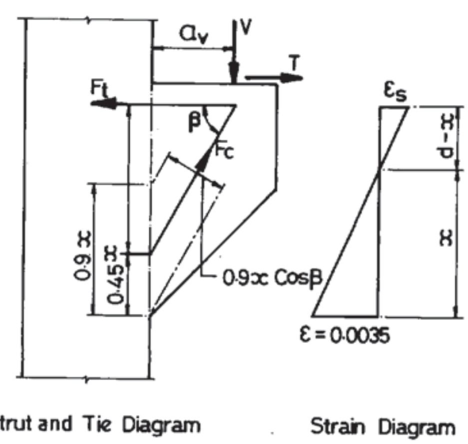
$$\frac{d}{R} = 1.0866k$$

$$\frac{z}{R} = \frac{d}{R} - \frac{\bar{x}}{R}$$

For a range of values of f_{cu}, k and p, the above equations can be solved for different values of e/R. Tables 11.18 to 11.27 give N/R^2 and z/R for different values of e/R.

Note: The above equations are valid up to $x = 1.111h = 2.22R$.

1.11 ULTIMATE LIMIT STATE – CORBELS



SK 1/22 Concrete corbel.

The derivation of Fig. 5.1

$$v = \frac{V}{bd}$$

From the strut and tie diagram:

$$F_t = T + F_c \cos\beta = T + \frac{Va_v}{z}$$

$$F_c = \left(\frac{0.67 f_{cu}}{1.5}\right) b\ 0.9x \cos\beta$$

$$= 0.402 f_{cu} b x \cos\beta$$

$$V = F_c \sin\beta$$

$$= 0.402 f_{cu} b x \cos\beta \sin\beta$$

$$\cos\beta = \frac{a_v}{(a_v^2 + z^2)^{\frac{1}{2}}}$$

$$\sin\beta = \frac{z}{(a_v^2 + z^2)^{\frac{1}{2}}}$$

$$v = \frac{V}{bd} = 0.402 f_{cu} x \frac{z a_v}{(a_v^2 + z^2) d}$$

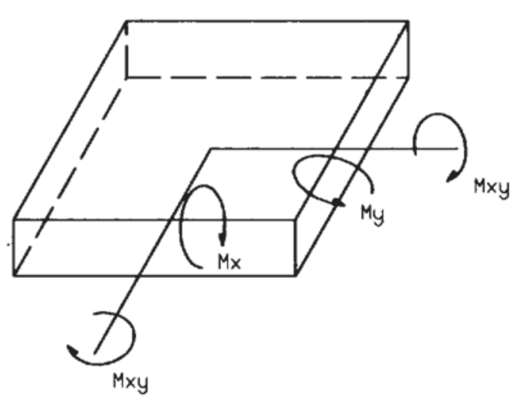
$$\frac{v}{f_{cu}} = 0.402\left(\frac{z}{d}\right)\left(\frac{x a_v}{a_v^2 + z^2}\right)$$

Substituting $x = (d - z)/0.45$,

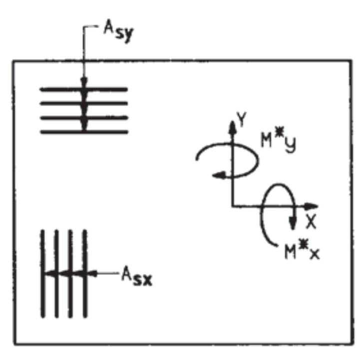
$$\frac{v}{f_{cu}} = \frac{0.893\left(\frac{z}{d}\right)\left(\frac{a_v}{d}\right)\left(1 - \frac{z}{d}\right)}{\left(\frac{a_v}{d}\right)^2 + \left(\frac{z}{d}\right)^2}$$

From the above equation the graphs in Fig. 5.1 have been drawn.

1.12 WOOD–ARMER COMBINATION OF MOMENT TRIADS



SK 1/23 Moment triad in a slab panel.



SK 1/24 Direction of orthogonal reinforcement in a slab panel.

Orthogonal reinforcement

Bottom steel

$$M_x^* = M_x + |M_{xy}|$$
$$M_y^* = M_y + |M_{xy}|$$

If $M_x^* < 0$, then $M_x^* = 0$

and $M_y^* = M_y + \left|\dfrac{M_{xy}^2}{M_x}\right|$

If $M_y^* < 0$, then $M_y^* = 0$

and $M_x^* = M_x + \left|\dfrac{M_{xy}^2}{M_y}\right|$

Top steel

$$M_x^* = M_x - |M_{xy}|$$
$$M_y^* = M_y - |M_{xy}|$$

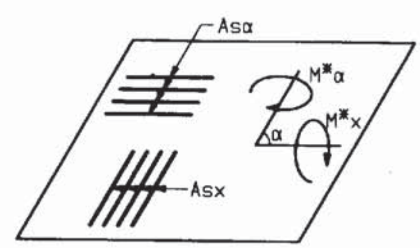
If $M_x^* > 0$, then $M_x^* = 0$

and $M_y^* = M_y - \left|\dfrac{M_{xy}^2}{M_x}\right|$

If $M_y^* > 0$, then $M_y^* = 0$

and $M_x^* = M_x - \left|\dfrac{M_{xy}^2}{M_y}\right|$

Skew reinforcement



SK 1/25 Direction of skew reinforcement in a slab panel.

Bottom steel

$$M_x^* = M_x + 2M_{xy}\cot\alpha + M_y\cot^2\alpha + \left|\frac{M_{xy} + M_y\cot\alpha}{\sin\alpha}\right|$$

$$M_\alpha^* = \frac{M_y}{\sin^2\alpha} + \left|\frac{M_{xy} + M_y\cot\alpha}{\sin\alpha}\right|$$

If $M_x^* < 0$, then $M_x^* = 0$

and $M^*_\alpha = \left(\frac{1}{\sin^2\alpha}\right)\left(M_y + \left|\frac{(M_{xy} + M_y\cot\alpha)^2}{M_x + 2M_{xy}\cot\alpha + M_y\cot^2\alpha}\right|\right)$

If $M^*_\alpha < 0$, then $M^*_\alpha = 0$

and $M^*_x = M_x + 2M_{xy}\cot\alpha + M_y\cot^2\alpha + \left|\frac{(M_{xy} + M_y\cot\alpha)^2}{M_y}\right|$

Top steel

$$M^*_x = M_x + 2M_{xy}\cot\alpha + M_y\cot^2\alpha - \left|\frac{M_{xy} + M_y\cot\alpha}{\sin\alpha}\right|$$

$$M^*_\alpha = \frac{M_y}{\sin^2\alpha} - \left|\frac{M_{xy} + M_y\cot\alpha}{\sin\alpha}\right|$$

If $M^*_x > 0$, then $M^*_x = 0$

and $M^*_\alpha = \left(\frac{1}{\sin^2\alpha}\right)\left(M_y - \left|\frac{(M_{xy} + M_y\cot\alpha)^2}{M_x + 2M_{xy}\cot\alpha + M_y\cot^2\alpha}\right|\right)$

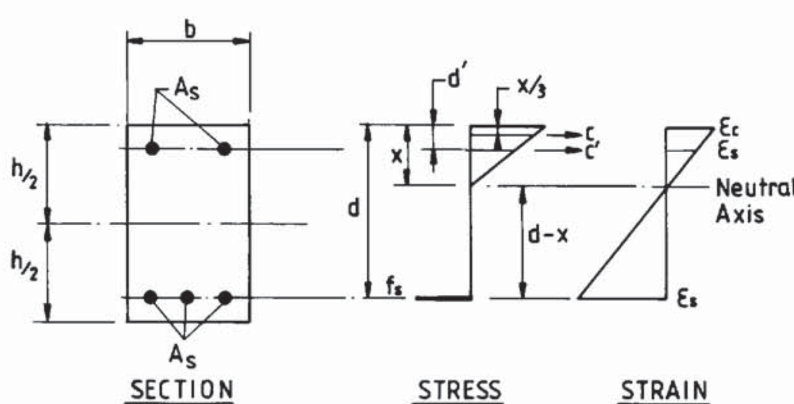
If $M^*_\alpha > 0$, then $M^*_\alpha = 0$

and $M^*_x = M_x + 2M_{xy}\cot\alpha + M_y\cot^2\alpha - \left|\frac{(M_{xy} + M_y\cot\alpha)^2}{M_y}\right|$

1.13 SERVICEABILITY LIMIT STATE – BENDING AND DIRECT LOADS

1.13.1 Serviceability limit state: uniaxial bending

C' = compressive force in bars in compression with allowance for area of concrete occupied by bars
C = compressive force in concrete stress block assumed triangular



SK 1/26 Serviceability uniaxial bending.

T = tensile force in bars in tension
M = applied bending moment
x = depth of neutral axis from compressive face
ε_c = strain in extreme compressive fibre of concrete
ε_s' = strain in compressive steel
ε_s = strain in tensile steel
$m = E_s/E_c$

From strain diagram,

$$\frac{\varepsilon_c}{\varepsilon_s'} = \frac{x}{x-d'} \quad \text{or} \quad \frac{f_c}{f_s'} = \frac{E_c}{E_s}\left(\frac{x}{d-d'}\right)$$

$$\frac{\varepsilon_c}{\varepsilon_s} = \frac{x}{x-d} \quad \text{or} \quad \frac{f_c}{f_s} = \frac{E_c}{E_s}\left(\frac{x}{d-x}\right)$$

$$C = 0.5bxf_c$$

$$C' = f_s'A_s' - f_cA_s'\left(\frac{x-d'}{x}\right)$$

$$= (m-1)f_cA_s'\left(\frac{x-d'}{x}\right)$$

$$T = f_sA_s = mf_cA_s\left(\frac{d-x}{x}\right)$$

Taking moment about the steel in tension,

$$C\left(d - \frac{x}{3}\right) + C'(d-d') = M$$

$$\text{or} \quad 0.5bxf_c\left(d - \frac{x}{3}\right) + (m-1)f_cA_s'\left(\frac{x-d'}{x}\right)(d-d') = M$$

Equating the loads on the section,

$$C + C' = T$$

$$\text{or} \quad 0.5bxf_c + (m-1)f_cA_s'\left(\frac{x-d'}{x}\right) = mf_cA_s\left(\frac{d-x}{x}\right)$$

Eliminating f_c and multiplying by $2x/bd^2$,

$$\frac{x^2}{d^2} + \frac{2(m-1)A_s'(x-d')}{bd^2} - \frac{2mA_s(d-x)}{bd^2} = 0$$

Simplifying, and substituting $p = A_s/bd$ and $p' = A_s'/bd$,

$$\left(\frac{x}{d}\right)^2 + 2[(m-1)p' + mp]\left(\frac{x}{d}\right) - 2\left[(m-1)p'\left(\frac{d'}{d}\right) + mp\right] = 0$$

$$\text{or} \quad \frac{x}{d} = \left\{\left[mp + (m-1)p'\right]^2 + 2\left[mp + (m-1)\left(\frac{d'}{d}\right)p'\right]\right\}^{\frac{1}{2}} - [mp + (m-1)p']$$

Put $p' = 0$, where compressive steel is not present.

Having found x using the above expression, find f_c.

$$f_c = \frac{M}{0.5bx(d - x/3) + (m - 1)A_s'(x - d')(d - d')/x}$$

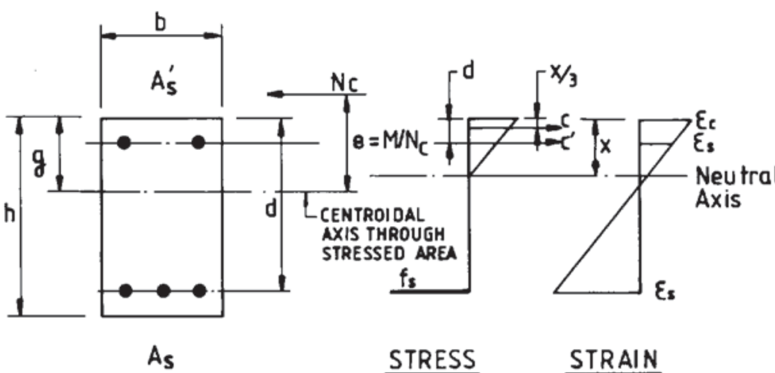
$$= \frac{M}{k_2bd^2 + k_3A_s'(d - d')}$$

where $k_2 = \left(\frac{x}{2d}\right)\left(1 - \frac{x}{3d}\right)$

$$k_3 = (m - 1)\left(1 - \frac{d'}{x}\right)$$

$$f_s = \text{tensile stress in steel} = mf_c\left(\frac{d}{x} - 1\right)$$

1.13.2 Serviceability limit state: uniaxial bending and compression



SK 1/27 Serviceability uniaxial bending and thrust.

Using the same symbols as in Section 1.13.1 and N_c is the compressive force, equating the loads on the section:

$$C' + C = N_c + T \quad \text{and} \quad e = \frac{M}{N_c}$$

The output from a computer program will give an axial compression of a member with a coacting moment. This axial compression theoretically for a reinforced concrete section may be considered as acting at the centroid of the stressed section. This will mean finding the centroid of the stressed area ignoring concrete in tension. On the other hand, the line of application of the compressive load may also be assumed to coincide with the centroid of the full concrete section ignoring any reinforcement.

The distance of the centroid of the stressed area from the compressive face of the rectangular section is called g and may be found by taking moments of all transformed areas of steel about the compressive face of the section

$$g = \frac{bx(x/2) + (m-1)A_s'd' + mA_sd}{bx + (m-1)A_s' + mA_s}$$

$g = \dfrac{h}{2}$ where the load is assumed to act at the centroid of the full concrete section ignoring steel.

(See Section 1.13.1 for expressions of C and C'.)

Taking moment about tension steel,

$$-N_c(e + d - g) + C'(d - d') + C\left(d - \frac{x}{3}\right) = 0$$

or $$-N_c(e + d - g) + (m-1)f_cA_s'\left(\frac{x - d'}{x}\right)(d - d')$$

$$+ 0.5bxf_c\left(d - \frac{x}{3}\right) = 0$$

or $$(m-1)\left(1 - \frac{d'}{x}\right)(d - d')\frac{f_cA_s'}{d} + 0.5bx\left(1 - \frac{x}{3d}\right)f_c$$

$$= N_c\left[\frac{(e-g)}{d} + 1\right]$$

$$k_1 = \left[\frac{(e-g)}{d} + 1\right]$$

$$k_2 = \left(\frac{x}{2d}\right)\left(1 - \frac{x}{3d}\right)$$

$$k_3 = (m-1)\left(1 - \frac{d'}{x}\right)$$

where k_1, k_2 and k_3 are non-dimensional constants.

$$k_3\left(1 - \frac{d'}{d}\right)f_cA_s' + k_2bdf_c = k_1N_c$$

or $$f_c = \frac{k_1N_c}{k_3\left(1 - \frac{d'}{d}\right)A_s' + k_2bd}$$

$$T = C' + C - N_c$$

or $$A_sf_s = (m-1)f_cA_s'\left(\frac{x - d'}{x}\right) + 0.5bxf_c - N_c$$

$$= k_3f_cA_s' + 0.5bxf_c - N_c$$

or $$f_s = \frac{f_c(k_3A_s' + 0.5bx) - N_c}{A_s}$$

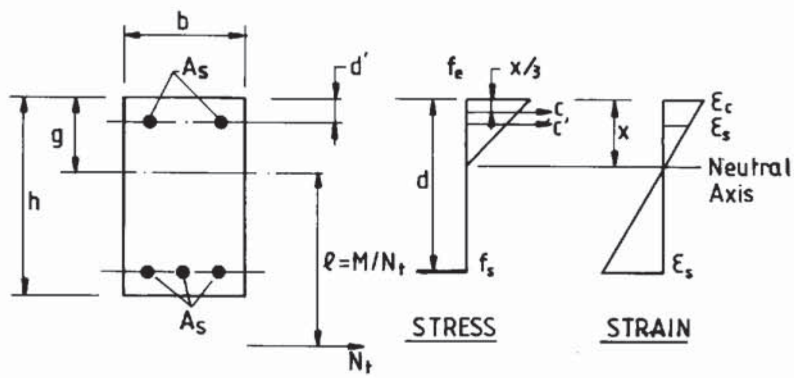
From strain diagram, see Section 1.13.1

$$\frac{f_s}{f_c} = \frac{m(d - x)}{x}$$

$$\text{or} \quad x = \frac{d}{\left(1 + \frac{f_s}{mf_c}\right)}$$

The procedure is to assume x and then calculate f_c and f_s, and then check x. Repeat this process until convergence is reached.

1.13.3 Serviceability limit state: uniaxial bending and tension



SK 1/28 Serviceability uniaxial bending and tension.

Using the same symbols as in Section 1.13.1 and N_t is the tensile force, equating the loads:

$$C' + C + N_t = T \quad \text{and} \quad e = \frac{M}{N_t}$$

The expressions for e, g, k_2 and k_3 are exactly the same as in Section 1.13.2 and g may be taken equal to $h/2$ where the point of application of the tensile load is at the centroid of the full concrete section ignoring steel. (See Section 1.13.1 for expressions of C and C'.)

Taking moment about tensile steel,

$$-N_t(e + g - d) + C'(d - d') + C\left(d - \frac{x}{3}\right) = 0$$

$$\text{or} \quad (m - 1)\left(1 - \frac{d'}{x}\right)(d - d')\frac{f_c A_s'}{d} + 0.5bx\left(1 - \frac{x}{3d}\right)f_c$$

$$= N_t\left[\frac{(e + g)}{d} - 1\right]$$

$$k_1 = \left[\frac{(e + g)}{d} - 1\right]$$

$$k_2 = \left(\frac{x}{2d}\right)\left(1 - \frac{x}{3d}\right)$$

$$k_3 = (m - 1)\left(1 - \frac{d'}{x}\right)$$

or $$f_c = \frac{k_1 N_t}{k_3\left(1 - \frac{d'}{d}\right)A'_s + k_2 bd}$$

$$A_s f_s = T = C' + C + N_t$$

or $$f_s = \frac{f_c(k_3 A'_s + 0.5bx) + N_t}{A_s}$$

$$x = \frac{d}{\left(1 + \frac{f_s}{mf_c}\right)}$$

as in Section 1.13.2. Check assumed value of x and repeat until convergence is reached.